Thinning, yield, and diameter distribution functions for loblolly pine in the Piedmont

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Contents

1 Introduction

The *LobDyn* growth model projects stand-level state variables: top height, basal area, trees per hectare. Other stand characteristics can be calculated (e.g., quadratic mean dbh or average spacing) or estimated as functions of the current state. Output functions, in dynamical systems jargon. Estimates for some common quantities of interest are obtained here. Some of these might be useful also in other applications.

2 Thinning

Often thinnings are specified in terms of residual or removed number of trees, and it may be necessary to estimate the corresponding basal areas. Or vice-versa. The relationships obtained here reflect the selection procedures used in the experimental plots, that may differ from typical operational conditions. Actual measurements should be used whenever possible.

Note that these are not really output functions, but may be viewed as part of the dynamics of the process of thinning.

The model is based on the following differential equation for the reduction in basal area (B, m^2/ha) relative to the reduction in number of trees (N, ha^{-1}):

$$\frac{\mathrm{d}\ln B}{\mathrm{d}\ln N} = aB^b N^c H^d$$

(García, 1984). H is top height (m), and a, b, c, d, are parameters to be estimated. In particular model versions any subset of the parameters b, c and d may be fixed at 0. If N_0 and B_0 are the values before thinning, integration predicts the basal area after thinning given N after thinning as

$$\ln B = -\ln[B_0^{-b} - \tfrac{ab}{c} H^d (N^c - N_0^c)]/b \tag{1}$$

if $b \neq 0$ and $c \neq 0$, or

$$\ln B = \ln B_0 + \tfrac{a}{c} H^d (N^c - N_0^c) \tag{2}$$

if $b = 0$ and $c \neq 0$. If $c = 0$, substitute $\frac{1}{c}(N^c - N_0^c) \to \ln(N/N_0)$. It is assumed that H does not change significantly.

Equations (1)–(2) were fitted for all the combinations of zero parameters by nonlinear least-squares, using all the recorded thinnings in the data (147 observations). Table 1 summarizes the best results for each number of free parameters. The tabulated values are the residual standard error, Akaike's information criterion, and Schwartz's Bayesian criterion.

The full model is best, with $a = 13.39$, $b = 0.4874$, $c = -0.3965$, and $d = -0.6541$. The equation to estimate the basal area after thinning is then:

$$B = [B_0^{-0.4874} - 16.46 H^{-0.6541} (N_0^{-0.3965} - N^{-0.3965})]^{-1/0.4874} \tag{3}$$

Table 1: Indices of fit for $\ln B$ regressions ($n = 147$)

Parameters	RSE	AIC	BIC
a	0.06959	-363.4	-357.4
a, c	0.06870	-366.2	-357.2
a, c, d	0.06851	-366.0	-354.0
a, b, c, d	0.06433	-383.5	-368.6

To estimate N when B is known, one might use the inverse of (3), especially if mathematical consistency is important. A statistically better estimate, however, is obtained from a regression for $\ln N$:

$$N = [N_0^{-0.4616} - 0.05037 H^{0.5816}(B_0^{-0.3722} - B^{-0.3722})]^{-1/0.4616} \ . \qquad (4)$$

3 Total volume

3.1 Model development

Volumes were computed with the tree volume equations from Amateis and Burkhart (1987) and Burkhart et al. (2004):

$$v_i = \max\{0, \ -0.09653 + 0.00210 d^2 h\}$$

and

$$v_o = 0.18658 + 0.00250 d^2 h \ ,$$

where v_i and v_o are tree volumes inside and outside bark, respectively. in cubic feet, d is tree dbh in inches, and h is tree height in feet. At each measurement, tree volumes were accumulated over all live pine trees, before and after thinning if applicable, divided by plot area, and converted to m^3/ha. This produced 1158 observations of stand volume V_i and V_o, together with top height H (m), number of trees per hectare N, and basal area B (m^2/ha). The quadratic mean dbh $D = 200\sqrt{\frac{B}{\pi N}}$ (cm) was also obtained.

Stand volume functions estimate volume per hectare in terms of other stand variables, a typical form being

$$V/B = b_0 + b_1 H \ .$$

(Beekhuis, 1966). Using the volume–basal area ratio on the left-hand side helps with the regression homoscedasticity assumption. The relationship is illustrated with our data in Figure 1.

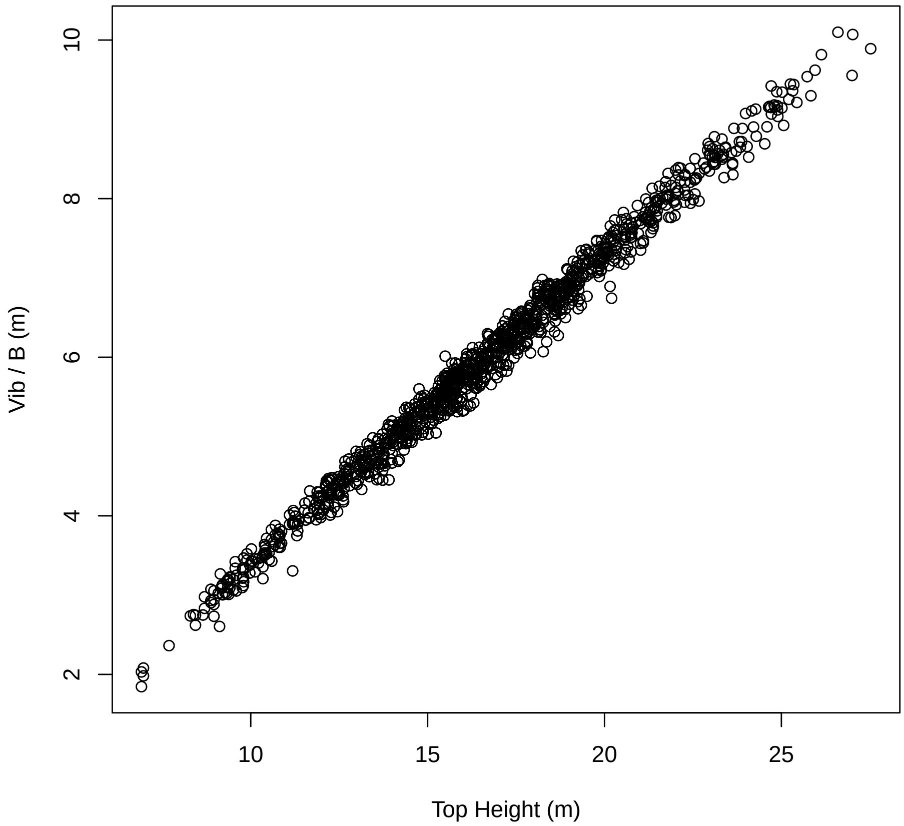

Figure 1: Volume–basal area ratio, inside bark.

As pointed out by Beekhuis (1966), however, the ratio can be expected to increase after a thinning. This is because the trees removed tend to be shorter than the rest of the stand, and therefore the mean height after thinning is larger than that before thinning, while the top height H does not change significantly. Dividing by H and graphing the form factor $V/(BH)$ in Figure 2 shows the effect more clearly. Including N in the regression can take this into account (García, 1984).

An initial (large) set of regressors was first selected by stepwise regression, using R's function `step` (R Development Core Team, 2009), starting with $H, N, B, D, \sqrt{N}$, their reciprocals, and all the cross-products among

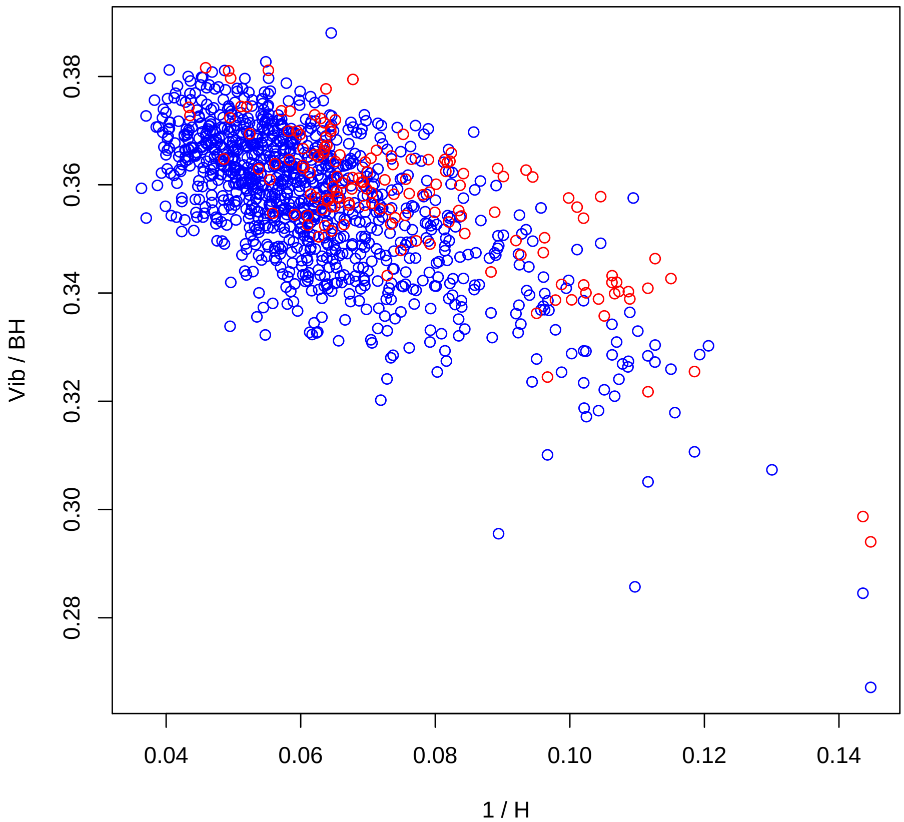

Figure 2: Inside-bark form factor. Measurements immediately after thinning are in red.

Table 2: Stand volume regressions ($n = 1158$)

Dependent	Right-hand side
V_i/B	$-0.5052 + 0.3894H$
V_i/B	$0.4093 + 0.4046H - 1.365H/D$
V_i/B	$-0.3905 + 0.3452H - 0.006818\sqrt{N} + 0.04237D$
V_o/B	$0.3206 + 0.428 > plot(I(Bafter - Bafter(Bbefore, Nbefore, Nafter, H))\ Bafter(Bbefore, Nbe$
V_o/B	$0.4044 + 0.4138H + 5.699D/N$
V_o/B	$-0.8055 + 0.4521H + 9.057/H + 5.480D/N$

these. Optimal subsets were then found with function `regsubsets` of package `leaps`. Table 2 lists the best regressions with up to four parameters. Additional variables continue improving the fit, and are nominally "statistically significant", but are probably not warranted in practice. Keeping in tune with the simplicity of the tree volume equations, I suggest using

$$V_i = \max\{0,\ B(0.4093 + 0.4046H - 1.365H/D)\}\ , \tag{5}$$

and

$$V_o = B(0.4044 + 0.4138H + 5.699D/N)\ . \tag{6}$$

For the *LobDyn* growth model, the thinning effect could be accounted for by including instead the relative closure R in the regression. Although perhaps more aesthetically pleasing, this alternative would be less convenient in other applications, requiring an estimate of R. This option has not been investigated.

3.2 Bark

Attempting to estimate bark from differences or ratios (e.g., percentages) between the inside bark and outside bark equations is likely to be unreliable. The tree volume equations are not necessarily compatible. They were developed independently, and their form may be too rigid to accommodate the differences. Figure 3 shows a bark factor computed dividing the outside and inside bark tree volume equation values, plotted over the range of observed $d^2 sav'vmodels.plth$ values. The ratios become very large for small trees.

Similar per-hectare ratios from (5) and (6) for all the plot measurements are graphed in Figure 4. Values for less than about 200 or 150 m^3/ha seem unrealistic. In some applications it may be preferable to use the inside bark equation (which is slightly more precise), and obtain outside bark volumes using an average bark factor.

3.3 Other tree volume equations

More recently, Tasissa et al. (1997) obtained new tree volume functions. The data was that from Amateis and Burkhart (1987), which proceeded from the establishment of the thinning trials, plus additional trees sampled from unthinned plots and from those marked for removal in the second thinning. The same *combined variable* equation form was used.

Tasissa et al. (1997) found statistically significant differences between the regressions for thinned and unthinned trees, and developed separate models.

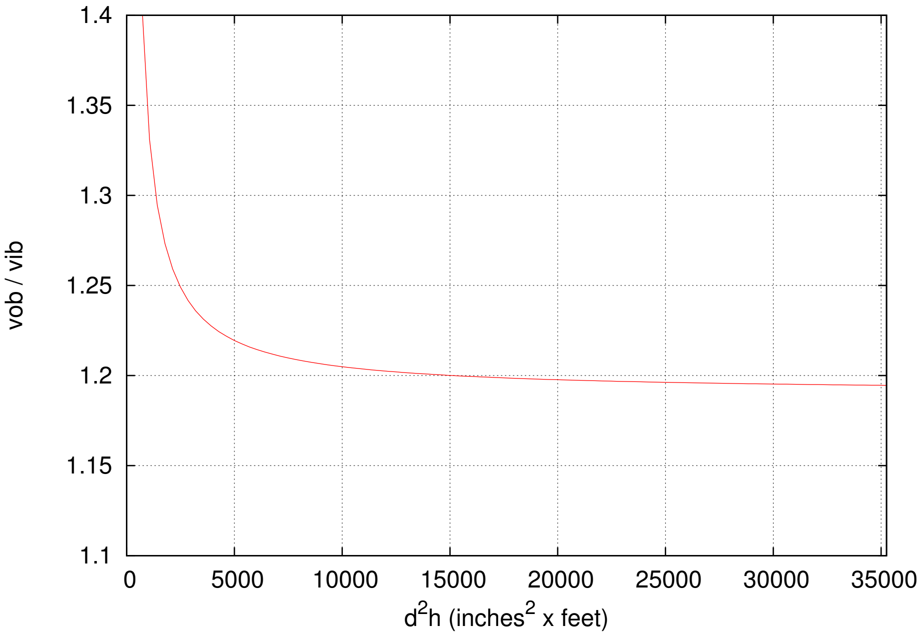

Figure 3: Outside bark to inside bark conversion factor implied by the tree volume equations.

For total volume inside bark, these are compared in Figure 5 with the one of Amateis and Burkhart (1987) used in Section 3.1.

These models were not used here for several reasons. An all-or-none thinned/unthinned indicator variable seems to me somewhat unsatisfactory, it implies differences between unthinned and lightly thinned, but not between light and heavy thinning. It does not match the principle of outputs being functions of the current state, that is, the volume depending of the tree directly observable physical characteristics. The treatment differences might be explained, for instance, by different height-diameter ratios. Their significance may disappear with something more flexible than the combined-variable equation, such as the *Australian formula* $v = \beta_0 + \beta_1 d^2 + \beta_2 h + \beta_3 d^2 h$.

4 Merchantable volumes

Burkhart et al. (2004), following Burkhart (1977), estimate merchantable

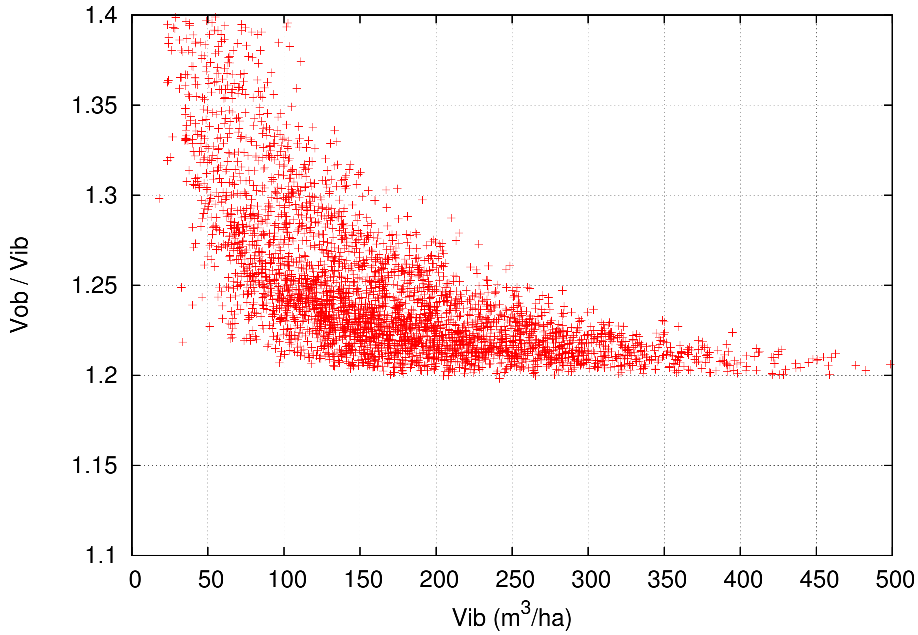

Figure 4: Per hectare bark conversion factors from the stand volume functions (5) and (6).

tree volumes to a limiting diameter λ as

$$m_i = v_i(1 - 0.527031\lambda^{3.14961}/d^{2.99580}) \tag{7}$$

and

$$m_o = v_o(1 - 0.458306\lambda^{3.22011}/d^{3.03262}) \;, \tag{8}$$

(converted to metric units). Here m_i and m_o are the tree merchantable volumes inside and outside bark, respectively, v_i and v_o are the corresponding total volumes, and d is the tree dbh. The volumes are in m^3, and the diameters in cm.

Adding over the trees in a plot and dividing by the plot area A, one obtains volumes per hectare, for instance for inside bark:

$$M_i = V_i(1 - 0.527031\lambda^{3.14961}\frac{1}{AV_i}\sum\frac{v_i}{d^{2.99580}}) \tag{9}$$

(ignoring trees where the values would be negative). We intend to obtain an

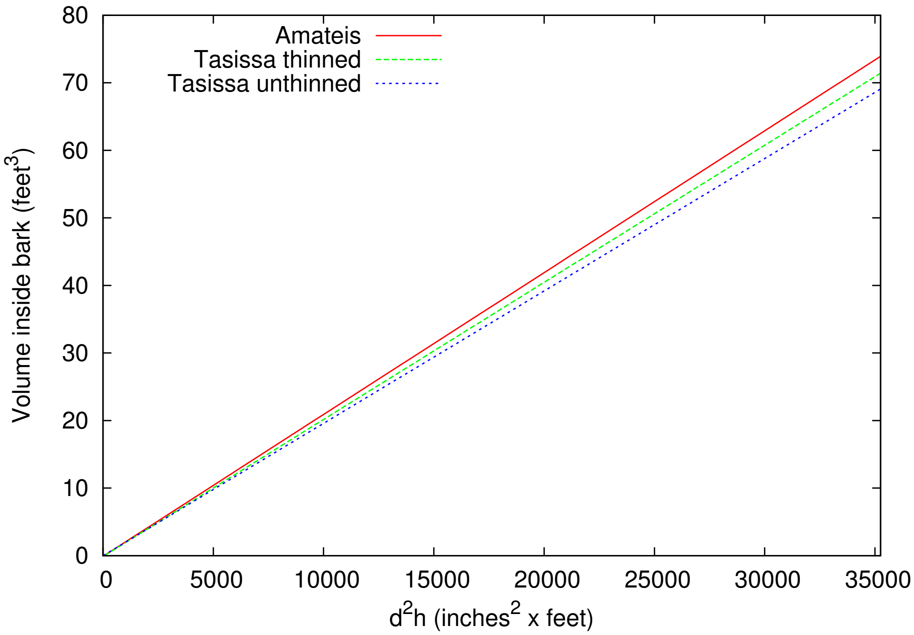

Figure 5: Comparison of tree volume functions

approximation of the form

$$\frac{1}{AV_i} \sum \frac{v_i}{d^{2.99580}} \approx \frac{\alpha}{D^\beta} \ ,$$

where D is the (quadratic) mean dbh, and thus end up with an equation of the same form as (7), but using stand-level variables.

To estimate α and β, I calculated the left-hand sides for each plot measurement, and used a log-transformed linear regression:

$$\ln(\frac{1}{AV_i} \sum \frac{v_i}{d^{2.99580}}) = \ln \alpha - \beta \ln D \ . \tag{10}$$

The data points are shown in Figure 6.

Results are given in Table 3 (one outlier was removed froml/softw/software.htm the outside-bark data). After substituting in (9) and the analogous equation

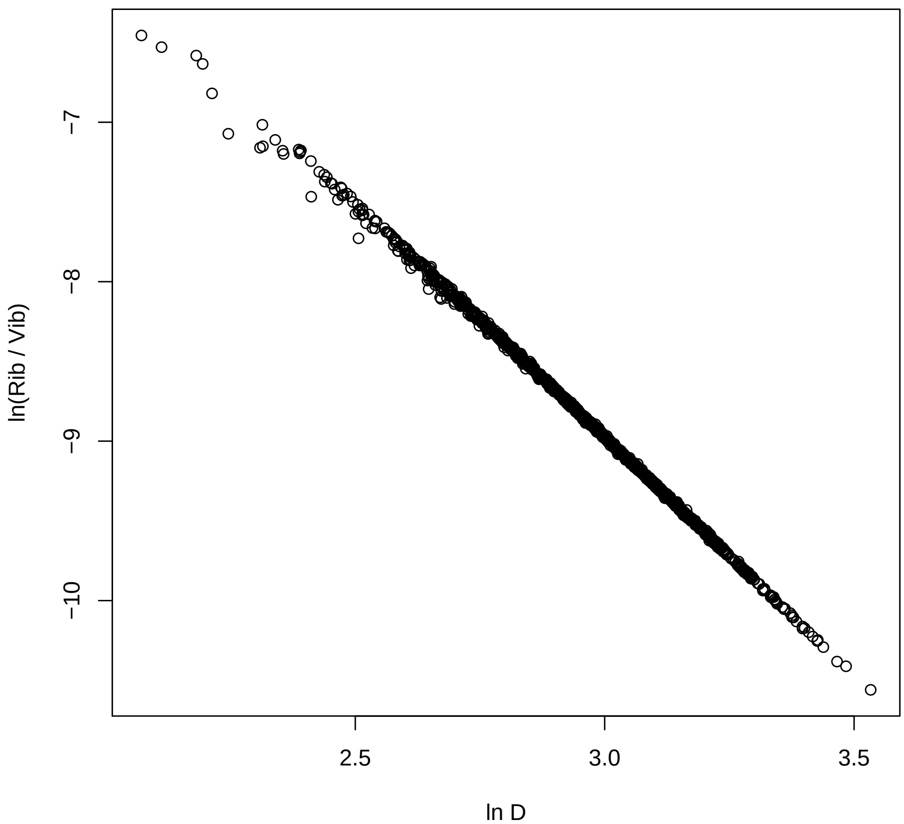

Figure 6: Data points for regression (10), inside bark.

Table 3: Regression (10) ($n = 1158, 1157$)

Volume	α	β	RSE	R^2
Inside bark	0.810365	2.92274	0.02295	0.999
Outside bark	1.63324	3.18397	0.06147	0.992

for over-bark,

$$M_i = V_i(1 - 0.4271\frac{\lambda^{3.150}}{D^{2.923}}) \tag{11}$$

$$M_o = V_o(1 - 0.7485\frac{\lambda^{3.220}}{D^{3.184}}) \tag{12}$$

5 Board-foot volumes

Burkhart et al. (2004) give the following board-foot tree volume equations, where d is the dbh in inches, and h is tree height in feet:

$$v_{\text{inter}} = \max\{0,\ -24.3816 + 0.005816(d^2h)^{1.0835}\}$$

for International 1/4-inch board-foot volume to a 6-inch top diameter.

$$v_{\text{doyle}} = 3.2492 + 0.00003386(d^2h)^{1.5651}$$

for Doyle board-foot volume to a 6-inch top diameter.

$$v_{\text{scrib}} = \max\{0,\ -29.7455 + 0.01888(d^2h)^{0.9521}\}$$

for Scribner board-foot volume to a 6-inch top diameter.

The same methods of Section 3 were used to obtain stand-level estimates. Regressing the unweighted volumes instead of volume ratios was also tried, but gave worse results. For some reason, a rather large number of regressors was needed to obtain good residual patterns. The chosen equations are:

$$\begin{aligned} V_{\text{inter}}/B =& \max\{0,\ 966.9 - 112.8D - 76.59H - 1996BH/N + \\ & 32553B/N + 8.321DH\} \quad (\text{RSE } 13.88,\ R^2\ 0.994) && (13) \\ V_{\text{doyle}}/B =& \max\{0,\ -550.7 + 28.14D + 4967/D + 281.7BH/N - \\ & 9802B/N + 0.2644DH\} \quad (\text{RSE } 4.383,\ R^2\ 0.996) && (14) \\ V_{\text{scrib}}/B =& \max\{0,\ 1579 - 177.9D - 2110BH/N + 54396B/N + \\ & 6.474DH - 731.4H/D\} \quad (\text{RSE } 13.75,\ R^2\ 0.992) && (15) \\ & && (16) \end{aligned}$$

with volumes in board feet per acre and predictors in metric units. Figure 7 shows the residuals for the International board-foot equation.

Important: The equations are not guaranteed to behave sensibly for stands younger than those represented in the data, less than 10 years-old or 7 m top height.

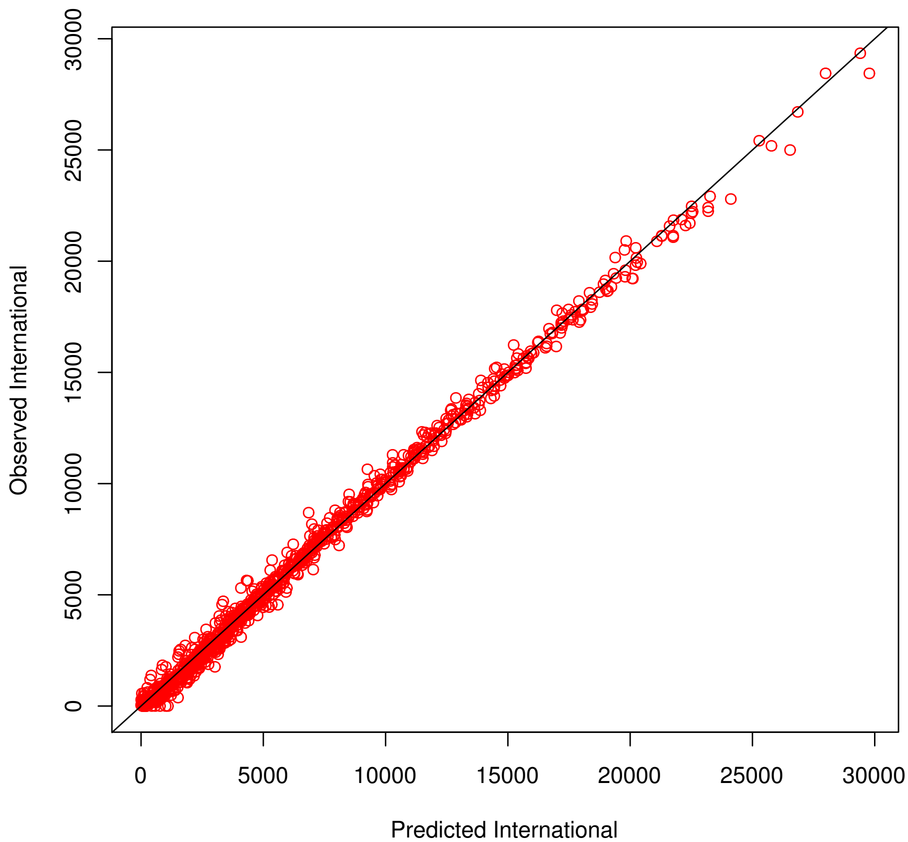

Figure 7: Residuals for the International board-foot stand volume equation.

6 Diameter distributions

For forest management it is useful to have estimates of tree size distributions. It must be realized, however, that spatial structure implies that a dbh distribution is essentially meaningless if not referred to a certain sampling area. In particular, distributions obtained from plot data can be very different from the distribution for a complete stand, which is what is wanted for management (García, 2006).

An unbiased estimate of the stand distribution is only possible with a random sample of several plots (García, 1992). In addition, moments higher

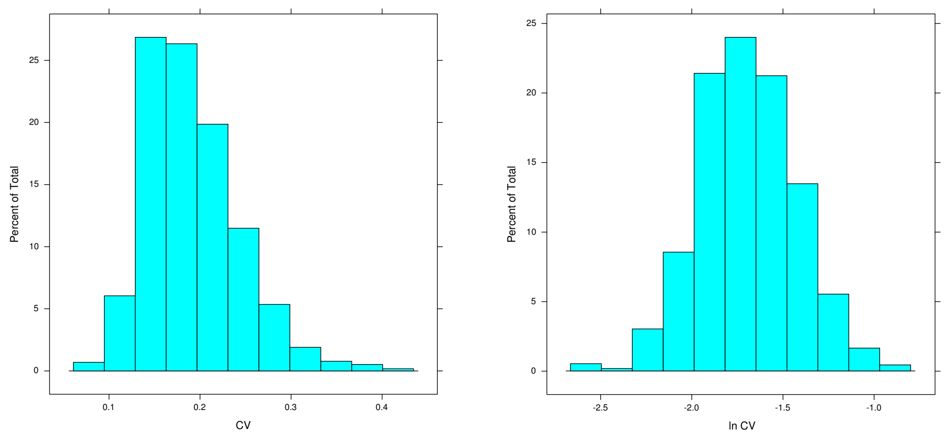

Figure 8: Histograms of the coefficient of variation of dbh and its logarithm, calculated for 1158 plot measurements.

than the mean have large sampling errors.

Because of these limitations, distributions should be considered only as rough approximations, and used with caution. Elaborate procedures are not warranted.

Here, the coefficient of variation (CV) and arithmetic mean dbh are estimated from stand-level state variables. They can be used to calculate parameters for a Weibull or other distribution. Histograms of the dbh CV from each plot measurement, and of its logarithm, are shown in Figure 6. A linear regression for the logarithm is used, because it is closer to normality, and it guarantees non-negative CV estimates. This is similar to the approach of García (1984), where tree basal area was used instead of dbh.

The chosen regression is

$$\ln \mathrm{CV} = -6.046 + 0.3702 \ln N + 0.5000 \ln H + 8.585/D$$

(RSE 0.1692, R^2 0.594), or

$$\mathrm{CV} = 0.002368 H^{0.5000} N^{0.3702} \mathrm{e}^{8.585/D} \; . \tag{17}$$

Note that this regression might not extrapolate well to stands younger than those represented in the data (age less than 10, or top height less than 7 m).

In terms of the quadratic mean D and arithmetic mean $\bar{d}$, the CV is

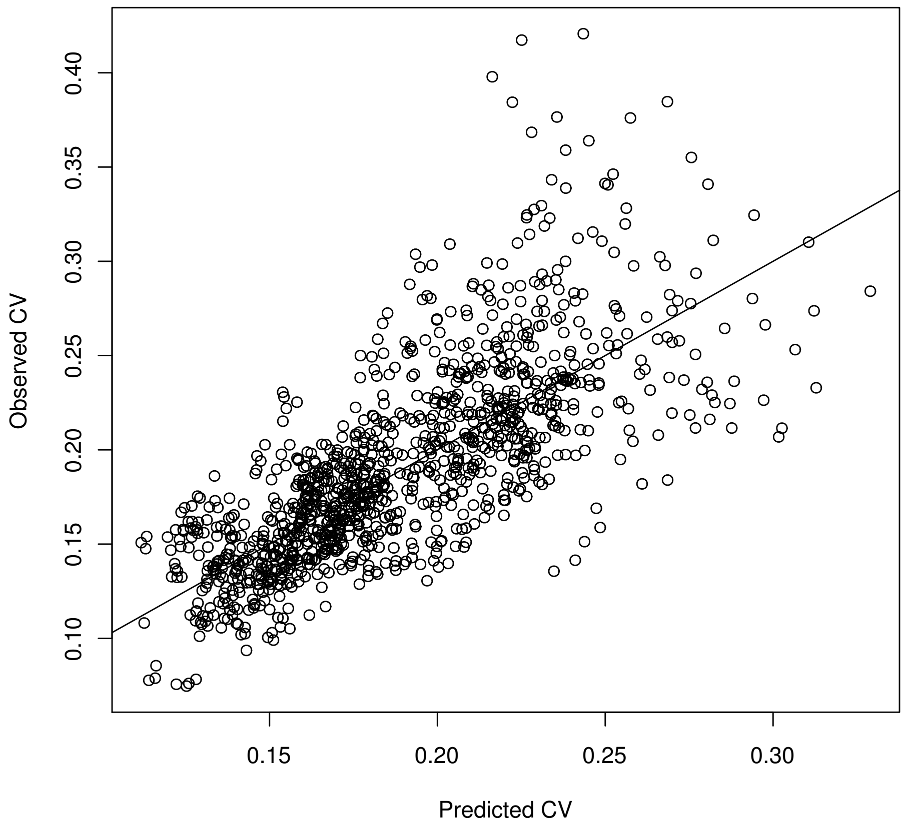

Figure 9: Residuals from the CV estimates (17).

$\sqrt{D^2 - \bar{d}^2}/\bar{d}$. Therefore, we estimate the arithmetic mean dbh as

$$\bar{d} = \frac{D}{\sqrt{1 + \mathrm{CV}^2}} \ . \tag{18}$$

A convenient and commonly used dbh distribution is the Weibull, with cumulative distribution function

$$F(x) = 1 - \mathrm{e}^{-(\frac{x}{a})^b} \ ,$$

and density

$$f(x) = \tfrac{b}{a}(\tfrac{x}{a})^{b-1}\mathrm{e}^{-(\frac{x}{a})^b} \ .$$

The CV is a function of the parameter b, but the equation cannot be solved analytically for b. García (1981) obtained approximations of various degrees of accuracy that avoid numerical iteration. The simplest one suffices here:

$$\begin{aligned}1/b =& \mathrm{CV}[1-(1-\mathrm{CV})^2(((0.0509991\mathrm{CV} \\ &- 0.117359)\mathrm{CV} - 0.00462251)\mathrm{CV} + 0.221016)\mathrm{CV}]\end{aligned} \tag{19}$$

Then, the parameter a can be calculated inverting the equation for the mean $\bar{d}$:

$$a = \bar{d}/\Gamma(1 + 1/b)\ . \tag{20}$$

If the gamma function is not available, it can be approximated by

$$\begin{aligned}\Gamma(1+1/b) \approx& ((((-0.1010678/b + 0.4245549)/b - 0.6998588)/b \\ &+ 0.9512363)/b - 0.5748646)/b + 1\ .\end{aligned}$$

Summarizing, one first obtains CV from (17), then $\bar{d}$ from (18), the Weibull b parameter from (19), and finally the a parameter from (20). Alternatively, the Weibull parameters can be obtained with software from `http://web.unbc.ca/~garcia/softw/software.htm#Weibullfitting`.

It is also common to use a 3-parameter Weibull with an additional location parameter c, substituting $x - c$ for x in the distribution. Figure 10 compares the density of the 2-parameter Weibull ($c = 0$) to the 3-parameter Weibull for various values of c, with a common mean and CV. Considering all the uncertainties, it seems clear that the differences are inconsequential in practice. In addition, (a) it is difficult to estimate c, it fails the regularity conditions on which methods such as maximum likelihood are based; (b) conveniently, if the dbh follows a 2-parameter Weibull with shape parameter b, the tree basal area is Weibull with shape parameter $b/2$, but the square or square root of a 3-parameter Weibull is not Weibull; (c) given any assumed positive lower limit c, it is usually possible to find a smaller tree somewhere in the forest, if one looks hard enough.

Similar observations can be made about alternative distribution functions. CV or variance estimation is unavoidably rough, because of the high variability of these parameters, and because of spatial correlations. Pretending to estimate higher moments or distribution shape seems unrealistic.

7 Green weight

Harvested loblilly pine is often commercialized by weight, and estimates may be required. Log weights vary widely depending on season, weather (affecting

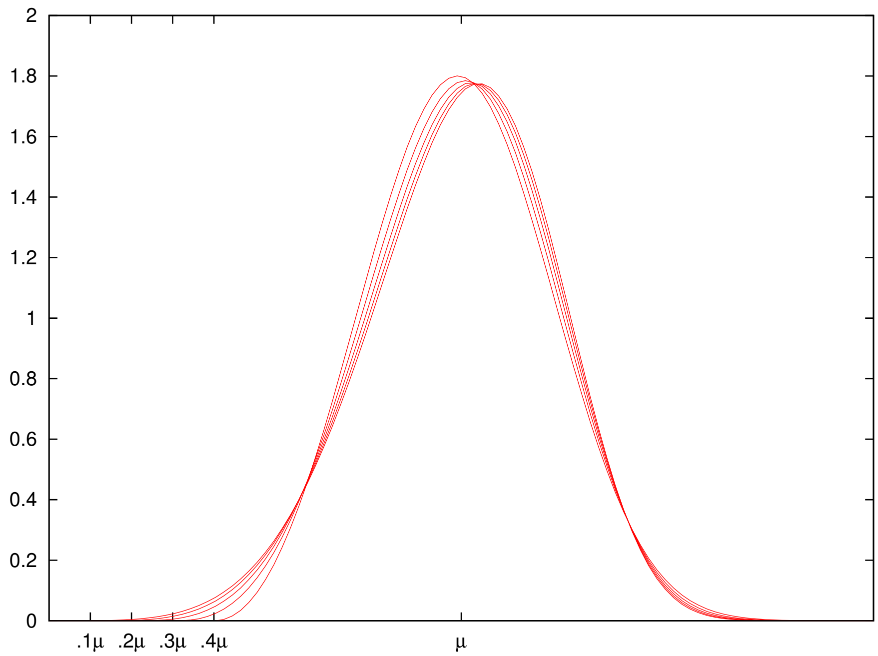

Figure 10: 3-parameter Weibull with CV = 0.2, mean μ, and $c = 0, 0.1\mu, 0.2\mu, 0.3\mu$, and 0.4μ.

mud, dirt and water content), piece size, and other variables (Ellis, 1984). For our purposes, an approximate volume to weight conversion factor seems sufficient.

Equation (1) in Bullock and Burkhart (2003) predicts tree weight from tree dbh (d) and height (h):

$$w = -4.3238 + 0.1397d^2h$$

(in US units). This can be solved for the average d^2h substituting the mean weights from their Table 1. Then, the outside-bark tree volume equation from Burkhart et al. (2004) (Section 3, above) gives the average volume. From there, volume to weight conversion factors can be obtained:

Location	Weight (lb)	Mean d^2h ($144 \times$ ft^3)	Mean Vob (ft^3)	Weight/Volume (lb / ft^3)	Weight/Volume (tonnes / m^3)
East Texas	213.0	1555.7	4.0757	52.3	0.837
Virginia	200.9	1469.0	3.8592	52.1	0.834
Georgia	274.2	1993.7	5.1709	53.0	0.849
Combined	248.1	1806.9	4.7038	52.7	0.845

These conversion factors may be considered as representative for clean logs, presumably mostly in dry weather.

We can check using the *delta method* with the means and standard deviations of d and h from Table 1 of Bullock and Burkhart (2003). Developing in Taylor series up to second order around the means,

$$d^2h \approx \bar{d}^2\bar{h} + 2\bar{d}\bar{h}(d-\bar{d}) + \bar{d}^2(h-\bar{h}) + \bar{h}(d-\bar{d})^2 + \bar{d}(d-\bar{d})(h-\bar{h}) \ .$$

Taking expectations,

$$\overline{d^2h} \approx \bar{d}^2\bar{h} + \bar{h}\sigma_d^2 + 2\bar{d}\sigma_d\sigma_h\rho \ .$$

The correlation ρ between height and dbh is not known, but reasonable guesses can be tried. The following table shows the estimated $\overline{d^2h}$ for $\rho = 0.7$ and $\rho = 0.8$, and the values of ρ that give the previously estimated values. Agreement seems reasonable, except for East Texas where there might be some mistake in the published table.

Location	$\widehat{d^2h}$ above	$\overline{d^2h}$ for $\rho = 0.7$	$\overline{d^2h}$ for $\rho = 0.8$	ρ for $\hat{d^2h}$
East Texas	1556	1370	1407	1.193
Virginia	1470	1452	1473	0.780
Georgia	1994	1992	2008	0.711
Combined	1807	1789	1812	0.778

8 Biomass and carbon

Forest Products Laboratory (1999) list for loblolly pine the oven-dry weight to green volume ratio (basic density) as 0.47 in Tables 4-3a and 15, or 0.48 in Table 4-9 (tonnes / m^3).

Wood carbon content is usually taken as 48% or 50%. Therefore, there are approximately 0.23 tonnes of C per m^3.

Carbon sequestration and emmisions are commonly expressed in tonnes of CO_2. The molecular weights 144 for CO_2 and 12 for C result in a conversion factor of 0.84 tonnes of CO_2 per m^3.

References

Amateis, R. L., Burkhart, H. E., 1987. Cubic-foot volume equations for loblolly pine trees in cutover site-prepared plantations. Southern Journal of Applied Forestry 11, 190–192.

Beekhuis, J., 1966. Prediction of yield and increment in *Pinus radiata* stands in New Zealand. Technical Paper 49, Forest Research Institute, NZ Forest Service, (http://web.unbc.ca/~garcia/misc/beekhuis66.pdf).

Bullock, B. P., Burkhart, H. E., 2003. Equations for predicting green weight of loblolly pine trees in the South. Southern Journal of Applied Forestry 27 (3), 153–159.

Burkhart, H. E., 1977. Cubic-foot volume of loblolly pine to any merchantable top limit. Southern Journal of Applied Forestry 1, 7–9.

Burkhart, H. E., Amateis, R. L., Westfall, J. A., Daniels, R. F., February 2004. PTAEDA3.1: Simulation of individual tree growth, stand development and economic evaluation in loblolly pine plantations. Tech. rep., Department of Forestry, Virginia Tech, Blacksburg, Virginia, (http://www.fw.vt.edu/g&y_coop/ptaeda3.1%20manual.pdf).

Ellis, J. C., 1984. Weight/volume conversion factors for logs. Technical Release 6(3), New Zealand Logging Industry Research Association.

Forest Products Laboratory, 1999. Wood handbook — Wood as an engineering material. General Technical Report FPL–GTR–113, U.S. Department of Agriculture, Forest Service, Forest Products Laboratory, Madison, WI:, 463p.

García, O., 1981. Simplified method-of-moments estimation for the Weibull distribution. New Zealand Journal of Forestry Science 11, 304–306.

García, O., 1984. New class of growth models for even-aged stands: *Pinus radiata* in Golden Downs forest. New Zealand Journal of Forestry Science 14, 65–88.

García, O., 1992. What is a diameter distribution? In: Minowa, M., Tsuyuki, S. (Eds.), Proceedings of the Symposium on Integrated Forest Management Information Systems — An International Symposium — October 13–18, 1991, Tsukuba, Japan. Japan Society of Forest Planning Press, pp. 11–29, (http://web.unbc.ca/~garcia/publ/japan.pdf).

García, O., 2006. Scale and spatial structure effects on tree size distributions: Implications for growth and yield modelling. Canadian Journal of Forest Research 36 (11), 2983–2993.

R Development Core Team, 2009. R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, ISBN 3-900051-07-0 (`http://www.R-project.org`).

Tasissa, G., Burkhart, H. E., Amateis, R. L., 1997. Volume and taper equations for thinned and unthinned loblolly pine trees in cutover, site-prepared plantations. Southern Journal of Applied Forestry 21 (3), 146–152.